

# NECESSARY AND SUFFICIENT CONDITIONS FOR SOLVABILITY OF INVERSE PROBLEM FOR DIRAC OPERATORS WITH DISCONTINUOUS COEFFICIENT

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**ABSTRACT.** In this work, a complete solution of the inverse spectral problem for a class of Dirac differential equations system is given by spectral data (eigenvalues and normalizing numbers). As a direct problem, the eigenvalue problem is solved: the asymptotic formulas of eigenvalues, eigenfunctions and normalizing numbers of problem are obtained, spectral data is defined by the sets of eigenvalues and normalizing numbers. The expansion formula with respect to eigenfunctions is obtained. Gelfand-Levitan-Marchenko equation is derived. The main theorem on necessary and sufficient conditions for the solvability of inverse spectral problem is proved and the algorithm of reconstruction of potential from spectral data is given.

## 1. INTRODUCTION

We consider the boundary value problem generated by Dirac differential equations system

$$By' + \Omega(x)y = \lambda\rho(x)y, \quad 0 < x < \pi \quad (1)$$

with boundary condition

$$U(y) := y_1(0) = 0, \quad (2)$$

$$V(y) := (\lambda + h_1)y_1(\pi) + h_2y_2(\pi) = 0,$$

where

$$B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \Omega(x) = \begin{pmatrix} p(x) & q(x) \\ q(x) & -p(x) \end{pmatrix}, \quad y = \begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix},$$

$p(x), q(x)$  are real valued functions in  $L_2(0, \pi)$ ,  $\lambda$  is a spectral parameter,

$$\rho(x) = \begin{cases} 1, & 0 \leq x \leq a, \\ \alpha, & a < x \leq \pi, \end{cases}$$

$1 \neq \alpha > 0$ ,  $h_1$  and  $h_2$  are real numbers,  $h_2 > 0$ .

In the finite interval, in the case of  $\rho(x) \equiv 1$  in the equation (1) and the potential function  $\Omega(x)$  is continuous, the solvability of inverse problem according to two spectra was obtained in [10] (in this work the criterion was obtained for two sequences of real numbers to be the spectra of two boundary value problems of Dirac operator) and according to one spectrum and normalizing numbers was given in [6]. The inverse problem contained spectral parameter in boundary condition by

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2010 *Mathematics Subject Classification.* 34A55, 34L40.

*Key words and phrases.* Dirac operator, eigenvalues and normalizing numbers, expansion formula, inverse problem, necessary and sufficient conditions.

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spectral function was studied in [16]. Inverse spectral problems for Dirac operator with summable potential were worked in [3, 21, 24]. Reconstruction of Dirac operator from nodal data was carried out in [29]. For Dirac operator with peculiarity inverse problem was found out in [23]. Using Weyl-Titchmarsh function, direct and inverse problems for Dirac type-system were studied in [8, 9, 25]. Solution of the inverse quasiperiodic problem for Dirac system was given in [22]. Inverse problem for weighted Dirac equations was investigated in [28]. For Dirac operator Ambarzumian-type theorems were proved in [12, 13, 30]. On a positive half line, inverse scattering problem for a system of Dirac equations of order  $2n$  was completely solved in [11] and when boundary condition contained spectral parameter, for Dirac operator, inverse scattering problem was worked in [2, 17]. The applications of Dirac differential equations system has been widespread in various areas of physics, such as [4, 5, 20, 26, 27] since Dirac equation was discovered to be associated with nonlinear wave equation in [1].

In this paper, as different from other studies, the boundary value problem (1), (2) has piecewise continuous coefficient, so the integral representation (not operator transformation) for the solution of equation (1) obtained in [14] is used.

This paper is organized as follows: In section 2, based on this integral representation, the asymptotic formulas of eigenvalues, eigenfunctions and normalizing numbers of the boundary value problem (1), (2) are investigated. The completeness theorem with respect to eigenfunctions is proved. The spectral expansion formula is obtained and Parseval equality is given. In section 3, the main equation namely Gelfand-Levitan-Marchenko type equation is derived. In section 4, we show that the boundary value problem (1), (2) can be uniquely determined from its eigenvalues and normalizing numbers. Finally in section 5, the solution of inverse problem is obtained. Let's express this more clearly. We can state the inverse problem for a system of Dirac equations in the following way: let  $\lambda_n$  and  $\alpha_n$ , ( $n \in \mathbb{Z}$ ) are respectively eigenvalues and normalizing numbers of boundary value problem (1), (2). Knowing the spectral data  $\{\lambda_n, \alpha_n\}$ , ( $n \in \mathbb{Z}$ ) to indicate a method of determining the potential  $\Omega(x)$  and to find necessary and sufficient conditions for  $\{\lambda_n, \alpha_n\}$ , ( $n \in \mathbb{Z}$ ) to be the spectral data of a problem (1),(2), for this, we derive differential equation, Parseval equality and boundary conditions. The main theorem on the necessary and sufficient conditions for the solvability of inverse problem is proved and then algorithm of the construction of the function  $\Omega(x)$  by spectral data is given. Note that throughout this paper, we use the following notation:  $\tilde{\phi}$  denotes the transposed matrix of  $\phi$ .

## 2. PRELIMINARIES

The inner product in Hilbert space  $H = L_{2,\rho}(0, \pi; \mathbb{C}^2) \oplus \mathbb{C}$  is defined by

$$\langle Y, Z \rangle := \int_0^\pi [y_1(x)\bar{z}_1(x) + y_2(x)\bar{z}_2(x)] \rho(x)dx + \frac{1}{h_2}y_3\bar{z}_3,$$

where

$$Y = \begin{pmatrix} y_1(x) \\ y_2(x) \\ y_3 \end{pmatrix} \in H, \quad Z = \begin{pmatrix} z_1(x) \\ z_2(x) \\ z_3 \end{pmatrix} \in H.$$

Let us define

$$L(Y) := \begin{pmatrix} l(y) \\ -h_1y_1(\pi) - h_2y_2(\pi) \end{pmatrix}$$

with

$$D(L) = \left\{ Y \mid Y = (y_1(x), y_2(x), y_3)^T \in H, \ y_1(x), \ y_2(x) \in AC[0, \pi], \ \right. \\ \left. y_3 = y_1(\pi), \ l(y) \in L_{2,\rho}(0, \pi; \mathbb{C}^2), \ y_1(0) = 0 \right\}$$

where

$$l(y) = \frac{1}{\rho(x)} \begin{pmatrix} y_2' + p(x)y_1 + q(x)y_2 \\ -y_1' + q(x)y_1 - p(x)y_2 \end{pmatrix}.$$

The boundary value problem (1), (2) is equivalent to equation  $LY = \lambda Y$ .

Let  $\varphi(x, \lambda)$  and  $\psi(x, \lambda)$  be solutions of the system (1) satisfying the initial conditions

$$\varphi(0, \lambda) = \begin{pmatrix} 0 \\ -1 \end{pmatrix}, \quad \psi(\pi, \lambda) = \begin{pmatrix} h_2 \\ -\lambda - h_1 \end{pmatrix}. \quad (3)$$

The solution  $\varphi(x, \lambda)$  has the following representation [14, 18]

$$\varphi(x, \lambda) = \varphi_0(x, \lambda) + \int_0^{\mu(x)} A(x, t) \begin{pmatrix} \sin \lambda t \\ -\cos \lambda t \end{pmatrix} dt, \quad (4)$$

where

$$\varphi_0(x, \lambda) = \begin{pmatrix} \sin \lambda \mu(x) \\ -\cos \lambda \mu(x) \end{pmatrix}, \quad \mu(x) = \begin{cases} x, & 0 \leq x \leq a, \\ \alpha x - \alpha a + a, & a < x \leq \pi, \end{cases}$$

$(A_{ij})_{i,j=1}^2$  is quadratic matrix function,  $A_{ij}(x, \cdot) \in L_2(0, \pi)$ ,  $i, j = 1, 2$  for fixed  $x \in [0, \pi]$  and  $A(x, t)$  is solution of the problem

$$BA_x'(x, t) + \rho(t)A_t'(x, t)B = -\Omega(x)A(x, t), \\ \Omega(x) = \rho(x)A(x, \mu(x))B - BA(x, \mu(x)), \quad (5) \\ A_{11}(x, 0) = A_{21}(x, 0) = 0.$$

The formula (5) gives the relation between the kernel  $A(x, t)$  and the coefficient of  $\Omega(x)$  of the equation (1).

The characteristic function  $\Delta(\lambda)$  of the problem  $L$  is

$$\Delta(\lambda) := W[\varphi(x, \lambda), \psi(x, \lambda)] = \varphi_2(x, \lambda)\psi_1(x, \lambda) - \varphi_1(x, \lambda)\psi_2(x, \lambda), \quad (6)$$

where  $W[\varphi(x, \lambda), \psi(x, \lambda)]$  is Wronskian of the solutions  $\varphi(x, \lambda)$  and  $\psi(x, \lambda)$  and independent of  $x \in [0, \pi]$ . The zeros of  $\Delta(\lambda)$  coincide with the eigenvalues  $\lambda_n$  of problem  $L$ . The functions  $\varphi(x, \lambda)$  and  $\psi(x, \lambda)$  are eigenfunctions and there exists a sequence  $\beta_n$  such that

$$\psi(x, \lambda_n) = \beta_n \varphi(x, \lambda_n), \quad \beta_n \neq 0. \quad (7)$$

Normalizing numbers are

$$\alpha_n := \int_0^\pi \left( |\varphi_1(x, \lambda_n)|^2 + |\varphi_2(x, \lambda_n)|^2 \right) \rho(x) dx + \frac{1}{h_2} |\varphi_1(\pi, \lambda_n)|^2.$$

The following relation holds [18]:

$$\dot{\Delta}(\lambda_n) = \beta_n \alpha_n, \quad (8)$$

where  $\dot{\Delta}(\lambda) = \frac{d}{d\lambda} \Delta(\lambda)$ .

**Theorem 1.** *i) The eigenvalues  $\lambda_n$ , ( $n \in \mathbb{Z}$ ) of boundary value problem (1), (2) are*

$$\lambda_n = \lambda_n^0 + \epsilon_n, \quad \{\epsilon_n\} \in l_2, \quad (9)$$

where  $\lambda_n^0 = \frac{n\pi}{\mu(\pi)}$  are zeros of function  $\lambda \sin \lambda \mu(\pi)$ . For the large  $n$ , the eigenvalues are simple.

*ii) The eigenfunctions of the boundary value problem can be represented in the form*

$$\varphi(x, \lambda_n) = \begin{pmatrix} \sin \frac{n\pi\mu(x)}{\mu(\pi)} \\ -\cos \frac{n\pi\mu(x)}{\mu(\pi)} \end{pmatrix} + \begin{pmatrix} \zeta_n^{(1)}(x) \\ \zeta_n^{(2)}(x) \end{pmatrix}, \quad (10)$$

where  $\sum_{n=-\infty}^{\infty} \left\{ \left| \zeta_n^{(1)}(x) \right|^2 + \left| \zeta_n^{(2)}(x) \right|^2 \right\} \leq C$ , in here  $C$  is a positive number.

*iii) Normalizing numbers of the problem (1), (2) are as follows*

$$\alpha_n = \mu(\pi) + \tau_n, \quad \{\tau_n\} \in l_2. \quad (11)$$

*Proof.* The proof of this theorem is similarly obtained in [18].  $\square$

We note that using (4), as  $|\lambda| \rightarrow \infty$  uniformly in  $x \in [0, \pi]$  the following asymptotic formulas are obtain:

$$\begin{aligned} \varphi_1(x, \lambda) &= \sin \lambda \mu(x) + O\left(\frac{1}{|\lambda|} e^{|Im \lambda| \mu(x)}\right), \\ \varphi_2(x, \lambda) &= -\cos \lambda \mu(x) + O\left(\frac{1}{|\lambda|} e^{|Im \lambda| \mu(x)}\right). \end{aligned} \quad (12)$$

Substituting the asymptotic formulas (12) into

$$\Delta(\lambda) = (\lambda + h_1) \varphi_1(\pi, \lambda) + h_2 \varphi_2(\pi, \lambda),$$

we get as  $|\lambda| \rightarrow \infty$

$$\Delta(\lambda) = \lambda \sin \lambda \mu(\pi) + O\left(e^{|Im \lambda| \mu(\pi)}\right). \quad (13)$$

**Proposition 2.** *The specification of the eigenvalues  $\lambda_n$ , ( $n \in \mathbb{Z}$ ) uniquely determines the characteristic function  $\Delta(\lambda)$  by formula*

$$\Delta(\lambda) = -\mu(\pi)(\lambda_0^2 - \lambda^2) \prod_{n=1}^{\infty} \frac{(\lambda_n^2 - \lambda^2)}{(\lambda_n^0)^2}. \quad (14)$$

*Proof.* Since the function  $\Delta(\lambda)$  is entire function, from Hadamard's theorem (see [15]), using (13) we obtain (14).  $\square$

**Theorem 3.** *a) The system of eigenfunctions  $\varphi(x, \lambda_n)$ , ( $n \in \mathbb{Z}$ ) of boundary value problem (1), (2) is complete in space  $L_{2,\rho}(0, \pi; \mathbb{C}^2)$ .*

*b) Let  $f(x) \in D(L)$ . Then the expansion formula*

$$\begin{aligned} f(x) &= \sum_{n=-\infty}^{\infty} a_n \varphi(x, \lambda_n), \\ a_n &= \frac{1}{\alpha_n} \langle f(x), \varphi(x, \lambda_n) \rangle \end{aligned} \quad (15)$$

is valid and the series converges uniformly with respect to  $x \in [0, \pi]$ . For  $f(x) \in L_{2,\rho}(0, \pi; \mathbb{C}^2)$  series (15) converges in  $L_{2,\rho}(0, \pi; \mathbb{C}^2)$ , moreover, Parseval equality holds

$$\|f\|^2 = \sum_{n=-\infty}^{\infty} \alpha_n |a_n|^2. \quad (16)$$

*Proof.* This theorem is analogously proved in [18].  $\square$

### 3. MAIN EQUATION

**Theorem 4.** For each fixed  $x \in (0, \pi]$  the kernel  $A(x, t)$  from the representation (4) satisfies the following equation

$$A(x, \mu(t)) + F(x, t) + \int_0^{\mu(x)} A(x, \xi) F_0(\xi, t) d\xi = 0, \quad 0 < t < x, \quad (17)$$

where

$$F_0(x, t) = \sum_{n=-\infty}^{\infty} \left[ \frac{1}{\alpha_n} \begin{pmatrix} \sin \lambda_n x \\ -\cos \lambda_n x \end{pmatrix} \tilde{\varphi}_0(t, \lambda_n) - \frac{1}{\mu(\pi)} \begin{pmatrix} \sin \lambda_n^0 x \\ -\cos \lambda_n^0 x \end{pmatrix} \tilde{\varphi}_0(t, \lambda_n^0) \right] \quad (18)$$

and

$$F(x, t) = F_0(\mu(x), t). \quad (19)$$

*Proof.* According to (4) we have,

$$\varphi_0(x, \lambda) = \varphi(x, \lambda) - \int_0^{\mu(x)} A(x, t) \begin{pmatrix} \sin \lambda t \\ -\cos \lambda t \end{pmatrix} dt. \quad (20)$$

It follows from (4) and (20) that

$$\begin{aligned} \sum_{n=-N}^N \frac{1}{\alpha_n} \varphi(x, \lambda_n) \tilde{\varphi}_0(t, \lambda_n) &= \sum_{n=-N}^N \frac{1}{\alpha_n} \varphi_0(x, \lambda_n) \tilde{\varphi}_0(t, \lambda_n) + \\ &+ \int_0^{\mu(x)} A(x, \xi) \left( \sum_{n=-N}^N \frac{1}{\alpha_n} \begin{pmatrix} \sin \lambda_n \xi \\ -\cos \lambda_n \xi \end{pmatrix} \tilde{\varphi}_0(t, \lambda_n) \right) d\xi \end{aligned}$$

and

$$\begin{aligned} \sum_{n=-N}^N \frac{1}{\alpha_n} \varphi(x, \lambda_n) \tilde{\varphi}_0(t, \lambda_n) &= \sum_{n=-N}^N \frac{1}{\alpha_n} \varphi(x, \lambda_n) \tilde{\varphi}(t, \lambda_n) - \\ &- \sum_{n=-N}^N \frac{1}{\alpha_n} \varphi(x, \lambda_n) \int_0^{\mu(t)} (\sin \lambda_n \xi, -\cos \lambda_n \xi) \tilde{A}(t, \xi) d\xi. \end{aligned}$$

Using the last two equalities, we obtain

$$\begin{aligned} &\sum_{n=-N}^N \left[ \frac{1}{\alpha_n} \varphi(x, \lambda_n) \tilde{\varphi}(t, \lambda_n) - \frac{1}{\mu(\pi)} \varphi(x, \lambda_n^0) \tilde{\varphi}(t, \lambda_n^0) \right] = \\ &= \sum_{n=-N}^N \left[ \frac{1}{\alpha_n} \varphi_0(x, \lambda_n) \tilde{\varphi}_0(t, \lambda_n) - \frac{1}{\mu(\pi)} \varphi_0(x, \lambda_n^0) \tilde{\varphi}_0(t, \lambda_n^0) \right] + \\ &+ \int_0^{\mu(x)} A(x, \xi) \sum_{n=-N}^N \left[ \frac{1}{\mu(\pi)} \begin{pmatrix} \sin \lambda_n^0 \xi \\ -\cos \lambda_n^0 \xi \end{pmatrix} \tilde{\varphi}_0(t, \lambda_n^0) \right] d\xi + \end{aligned}$$

$$\begin{aligned}
& + \int_0^{\mu(x)} A(x, \xi) \sum_{n=-N}^N \left[ \frac{1}{\alpha_n} \begin{pmatrix} \sin \lambda_n \xi \\ -\cos \lambda_n \xi \end{pmatrix} \tilde{\varphi}_0(t, \lambda_n) - \right. \\
& \quad \left. - \frac{1}{\mu(\pi)} \begin{pmatrix} \sin \lambda_n^0 \xi \\ -\cos \lambda_n^0 \xi \end{pmatrix} \tilde{\varphi}_0(t, \lambda_n^0) \right] d\xi + \\
& + \sum_{n=-N}^N \frac{1}{\alpha_n} \varphi(x, \lambda_n) \int_0^{\mu(t)} (\sin \lambda_n \xi, -\cos \lambda_n \xi) \tilde{A}(t, \xi) d\xi
\end{aligned}$$

or

$$\Phi_N(x, t) = I_{N1}(x, t) + I_{N2}(x, t) + I_{N3}(x, t) + I_{N4}(x, t), \quad (21)$$

where

$$\begin{aligned}
\Phi_N(x, t) &= \sum_{n=-N}^N \left[ \frac{1}{\alpha_n} \varphi(x, \lambda_n) \tilde{\varphi}_0(t, \lambda_n) - \frac{1}{\mu(\pi)} \varphi(x, \lambda_n^0) \tilde{\varphi}_0(t, \lambda_n^0) \right], \\
I_{N1}(x, t) &= \sum_{n=-N}^N \left[ \frac{1}{\alpha_n} \varphi_0(x, \lambda_n) \tilde{\varphi}_0(t, \lambda_n) - \frac{1}{\mu(\pi)} \varphi_0(x, \lambda_n^0) \tilde{\varphi}_0(t, \lambda_n^0) \right], \\
I_{N2}(x, t) &= \int_0^{\mu(x)} A(x, \xi) \sum_{n=-N}^N \left[ \frac{1}{\mu(\pi)} \begin{pmatrix} \sin \lambda_n^0 \xi \\ -\cos \lambda_n^0 \xi \end{pmatrix} \tilde{\varphi}_0(t, \lambda_n^0) \right] d\xi, \\
I_{N3}(x, t) &= \int_0^{\mu(x)} A(x, \xi) \sum_{n=-N}^N \left[ \frac{1}{\alpha_n} \begin{pmatrix} \sin \lambda_n \xi \\ -\cos \lambda_n \xi \end{pmatrix} \tilde{\varphi}_0(t, \lambda_n) - \right. \\
& \quad \left. - \frac{1}{\mu(\pi)} \begin{pmatrix} \sin \lambda_n^0 \xi \\ -\cos \lambda_n^0 \xi \end{pmatrix} \tilde{\varphi}_0(t, \lambda_n^0) \right] d\xi, \\
I_{N4}(x, t) &= \sum_{n=-N}^N \frac{1}{\alpha_n} \varphi(x, \lambda_n) \int_0^{\mu(t)} (\sin \lambda_n \xi, -\cos \lambda_n \xi) \tilde{A}(t, \xi) d\xi.
\end{aligned}$$

It is easily found by using (18) and (19) that

$$F(x, t) = \sum_{n=-\infty}^{\infty} \left[ \frac{1}{\alpha_n} \varphi_0(x, \lambda_n) \tilde{\varphi}_0(t, \lambda_n) - \frac{1}{\mu(\pi)} \varphi_0(x, \lambda_n^0) \tilde{\varphi}_0(t, \lambda_n^0) \right]. \quad (22)$$

Let  $f(x) \in AC[0, \pi]$ . Then according to expansion formula (15) in Theorem 3, we obtain uniformly on  $x \in [0, \pi]$

$$\lim_{N \rightarrow \infty} \int_0^\pi \Phi_N(x, t) f(t) \rho(t) dt = \sum_{n=-\infty}^{\infty} a_n \varphi(x, \lambda_n) - \sum_{n=-\infty}^{\infty} a_n^0 \varphi(x, \lambda_n^0) = 0. \quad (23)$$

From (22), we find

$$\begin{aligned}
& \lim_{N \rightarrow \infty} \int_0^\pi I_{N1}(x, t) f(t) \rho(t) dt = \\
& = \lim_{N \rightarrow \infty} \int_0^\pi \sum_{n=-N}^N \left[ \frac{1}{\alpha_n} \varphi_0(x, \lambda_n) \tilde{\varphi}_0(t, \lambda_n) - \frac{1}{\mu(\pi)} \varphi_0(x, \lambda_n^0) \tilde{\varphi}_0(t, \lambda_n^0) \right] f(t) \rho(t) dt \\
& = \int_0^\pi F(x, t) f(t) \rho(t) dt.
\end{aligned} \quad (24)$$

It follows from (4) that

$$\begin{pmatrix} \sin \lambda \xi \\ -\cos \lambda \xi \end{pmatrix} = \begin{cases} \varphi_0(\xi, \lambda), & \xi < a, \\ \varphi_0\left(\frac{\xi}{\alpha} + a - \frac{a}{\alpha}, \lambda\right), & \xi > a. \end{cases} \quad (25)$$

Taking into account (25) and expansion formula (15) in Theorem 3, we get

$$\begin{aligned} & \lim_{N \rightarrow \infty} \int_0^\pi I_{N2}(x, t) f(t) \rho(t) dt = \\ &= \int_0^\pi \left[ \int_0^{\mu(x)} A(x, \xi) \sum_{n=-N}^N \left[ \frac{1}{\mu(\pi)} \begin{pmatrix} \sin \lambda_n^0 \xi \\ -\cos \lambda_n^0 \xi \end{pmatrix} \tilde{\varphi}_0(t, \lambda_n^0) \right] d\xi \right] f(t) \rho(t) dt \\ &= \int_0^\pi \left[ \int_0^a A(x, \xi) \sum_{n=-\infty}^\infty \frac{1}{\mu(\pi)} \varphi_0(\xi, \lambda_n^0) \tilde{\varphi}_0(x, \lambda_n^0) d\xi \right] f(t) \rho(t) dt + \\ &+ \int_0^\pi \left[ \int_a^{\alpha x - \alpha a + a} A(x, \xi) \sum_{n=-\infty}^\infty \frac{1}{\mu(\pi)} \varphi_0\left(\frac{\xi}{\alpha} + a - \frac{a}{\alpha}, \lambda_n^0\right) \tilde{\varphi}_0(x, \lambda_n^0) d\xi \right] f(t) \rho(t) dt \\ &= \int_0^a A(x, \xi) f(\xi) d\xi + \int_a^{\alpha x - \alpha a + a} A(x, \xi) f\left(\frac{\xi}{\alpha} + a - \frac{a}{\alpha}\right) d\xi. \end{aligned}$$

Substituting  $\frac{\xi}{\alpha} + a - \frac{a}{\alpha} \rightarrow \xi'$ , we obtain

$$\begin{aligned} & \lim_{N \rightarrow \infty} \int_0^\pi I_{N2}(x, t) f(t) \rho(t) dt = \\ &= \int_0^a A(x, \xi) f(\xi) d\xi + \alpha \int_a^x A(x, \alpha \xi' - \alpha a + a) f(\xi') d\xi' \\ &= \int_0^a A(x, t) f(t) dt + \alpha \int_a^x A(x, \alpha t - \alpha a + a) f(t) dt \\ &= \int_0^x A(x, \mu(t)) f(t) \rho(t) dt. \end{aligned} \quad (26)$$

Now, we calculate

$$\begin{aligned} & \lim_{N \rightarrow \infty} \int_0^\pi I_{N3}(x, t) f(t) \rho(t) dt = \\ &= \lim_{N \rightarrow \infty} \int_0^\pi \int_0^{\mu(x)} A(x, \xi) \sum_{n=-N}^N \left[ \frac{1}{\alpha_n} \begin{pmatrix} \sin \lambda_n \xi \\ -\cos \lambda_n \xi \end{pmatrix} \tilde{\varphi}_0(t, \lambda_n) - \right. \\ &\quad \left. - \frac{1}{\mu(\pi)} \begin{pmatrix} \sin \lambda_n^0 \xi \\ -\cos \lambda_n^0 \xi \end{pmatrix} \tilde{\varphi}_0(t, \lambda_n^0) \right] f(t) \rho(t) d\xi dt \\ &= \int_0^\pi \left[ \int_0^{\mu(x)} A(x, \xi) F_0(\xi, t) d\xi \right] f(t) \rho(t) dt. \end{aligned} \quad (27)$$

Using (7), (8) and residue theorem, we get

$$\begin{aligned} & \lim_{N \rightarrow \infty} \int_0^\pi I_{N4}(x, t) f(t) \rho(t) dt = \\ &= \lim_{N \rightarrow \infty} \int_0^\pi \left[ \sum_{n=-N}^N \frac{1}{\alpha_n} \varphi(x, \lambda_n) \int_0^{\mu(t)} (\sin \lambda_n \xi, -\cos \lambda_n \xi) \tilde{A}(t, \xi) d\xi \right] f(t) \rho(t) dt \end{aligned}$$

$$\begin{aligned}
&= \lim_{N \rightarrow \infty} \int_0^\pi \left[ \sum_{n=-N}^N \frac{\psi(x, \lambda_n)}{\Delta(\lambda_n)} \int_0^{\mu(t)} (\sin \lambda_n \xi, -\cos \lambda_n \xi) \tilde{A}(t, \xi) d\xi \right] f(t) \rho(t) dt \\
&= \lim_{N \rightarrow \infty} \int_0^\pi \left[ \sum_{n=-N}^N \operatorname{Res}_{\lambda=\lambda_n} \frac{\psi(x, \lambda)}{\Delta(\lambda)} \int_0^{\mu(t)} (\sin \lambda \xi, -\cos \lambda \xi) \tilde{A}(t, \xi) d\xi \right] f(t) \rho(t) dt \\
&= \lim_{N \rightarrow \infty} \int_0^\pi \left[ \frac{1}{2\pi i} \int_{\Gamma_N} \frac{\psi(x, \lambda)}{\Delta(\lambda)} \int_0^{\mu(t)} (\sin \lambda \xi, -\cos \lambda \xi) \tilde{A}(t, \xi) d\xi d\lambda \right] f(t) \rho(t) dt \\
&= \lim_{N \rightarrow \infty} \int_0^\pi \left[ \frac{1}{2\pi i} \int_{\Gamma_N} \frac{\psi(x, \lambda)}{\Delta(\lambda)} e^{|Im \lambda| \mu(t)} \times \right. \\
&\quad \left. \times e^{-|Im \lambda| \mu(t)} \int_0^{\mu(t)} (\sin \lambda \xi, -\cos \lambda \xi) \tilde{A}(t, \xi) d\xi d\lambda \right] f(t) \rho(t) dt, \tag{28}
\end{aligned}$$

where  $\Gamma_N = \left\{ \lambda : |\lambda| = \lambda_N^0 + \frac{\pi}{2\mu(\pi)} \right\}$  is a oriented counter-clockwise,  $N$  is sufficiently large number. Taking into account, the asymptotic formulas as  $|\lambda| \rightarrow \infty$

$$\psi_1(x, \lambda) = h_2 \cos \lambda(\mu(\pi) - \mu(x)) - (\lambda + h_1) \sin \lambda(\mu(\pi) - \mu(x)) + O\left(e^{|Im \lambda|(\mu(\pi) - \mu(x))}\right),$$

$$\psi_2(x, \lambda) = -h_2 \sin \lambda(\mu(\pi) - \mu(x)) - (\lambda + h_1) \cos \lambda(\mu(\pi) - \mu(x)) + O\left(e^{|Im \lambda|(\mu(\pi) - \mu(x))}\right)$$

and the relations ([19], Lemma 1.3.1)

$$\lim_{|\lambda| \rightarrow \infty} \max_{0 \leq t \leq \pi} e^{-|Im \lambda| \mu(t)} \left| \int_0^{\mu(t)} A_{i,1}(t, \xi) \sin \lambda \xi d\xi \right| = 0,$$

$$\lim_{|\lambda| \rightarrow \infty} \max_{0 \leq t \leq \pi} e^{-|Im \lambda| \mu(t)} \left| \int_0^{\mu(t)} A_{i,2}(t, \xi) \cos \lambda \xi d\xi \right| = 0, \quad i = 1, 2,$$

it follows from (9) and (28) that

$$\lim_{N \rightarrow \infty} \int_0^\pi I_{N4}(x, t) f(t) \rho(t) dt = 0. \tag{29}$$

Thus, using (21), (23), (24), (26) (27) and (29), we find

$$\begin{aligned}
&\int_0^x A(x, \mu(t)) f(t) \rho(t) dt + \int_0^\pi F(x, t) f(t) \rho(t) dt + \\
&+ \int_0^\pi \left[ \int_0^{\mu(x)} A(x, \xi) F_0(\xi, t) d\xi \right] f(t) \rho(t) dt = 0.
\end{aligned}$$

Since  $f(x)$  can be chosen arbitrarily,

$$A(x, \mu(t)) + F(x, t) + \int_0^{\mu(x)} A(x, \xi) F_0(\xi, t) d\xi = 0, \quad 0 < t < x$$

is obtained.  $\square$



## 4. THEOREM FOR THE SOLUTION OF THE INVERSE PROBLEM

**Lemma 5.** *For each fixed  $x \in (0, \pi]$  the equation (17) has a unique solution  $A(x, \cdot) \in L_2(0, \mu(x))$ .*

*Proof.* When  $a < x$ , the equation (17) can be rewritten as

$$L_x A(x, \cdot) + K_x A(x, \cdot) = -F(x, \cdot),$$

where

$$(L_x f)(t) = \begin{cases} f(t), & t \leq a < x, \\ f(\alpha t - \alpha a + a), & a < t \leq x, \end{cases} \quad (30)$$

$$(K_x f) = \int_0^{\alpha x - \alpha a + a} f(\xi) F_0(\xi, t) d\xi, \quad 0 < t < x.$$

Now, we shall prove that  $L_x$  is invertible, i.e has a bounded inverse in  $L_2(0, \pi)$ .

Consider the equation  $(L_x f)(t) = \phi(t)$ ,  $\phi(t) \in L_2(0, \pi; \mathbb{C}^2)$ . From here and (30),

$$f(t) = (L_x^{-1} \phi)(t) = \begin{cases} \phi(t), & t \leq a, \\ \phi\left(\frac{t + \alpha a - a}{\alpha}\right), & a < t. \end{cases}$$

We show that

$$\|f\|_{L_2} = \|L_x^{-1} \phi\|_{L_2} \leq C \|\phi\|_{L_2}.$$

In fact,

$$\begin{aligned} & \int_0^\pi (|f_1(t)|^2 + |f_2(t)|^2) dt = \int_0^a (|\phi_1(t)|^2 + |\phi_2(t)|^2) dt + \\ & + \int_a^\pi \left( \left| \phi_1\left(\frac{t + \alpha a - a}{\alpha}\right) \right|^2 + \left| \phi_2\left(\frac{t + \alpha a - a}{\alpha}\right) \right|^2 \right) dt \\ & = \int_0^a (|\phi_1(t)|^2 + |\phi_2(t)|^2) dt + \alpha \int_a^{\frac{\pi + \alpha a - a}{\alpha}} (|\phi_1(t)|^2 + |\phi_2(t)|^2) dt \\ & \leq C \int_0^\pi (|\phi_1(t)|^2 + |\phi_2(t)|^2) dt. \end{aligned}$$

Thus, the operator  $L_x$  is invertible in  $L_2(0, \pi)$ . Therefore, the main equation (17) is equivalent to

$$A(x, \cdot) + L_x^{-1} K_x A(x, \cdot) = -L_x^{-1} F(x, \cdot)$$

and  $L_x^{-1} K_x$  is completely continuous in  $L_2(0, \pi)$ . Then it is sufficient to prove that the equation

$$g(\mu(t)) + \int_0^{\mu(x)} g(\xi) F_0(\xi, t) d\xi = 0 \quad (31)$$

has only trivial solution  $g(t) = 0$ . Let  $g(t)$  be a non-trivial solution of (31). Then

$$\int_0^x (g_1^2(\mu(t)) + g_2^2(\mu(t))) \rho(t) dt + \int_0^x \int_0^{\mu(x)} (g(\xi) F_0(\xi, t), g(\mu(t))) \rho(t) d\xi dt = 0.$$

It follows from (18) that

$$\begin{aligned} & \int_0^x (g_1^2(\mu(t)) + g_2^2(\mu(t))) \rho(t) dt + \\ & + \int_0^x \tilde{g}(\mu(t)) \rho(t) \int_0^{\mu(x)} g(\xi) \left( \sum_{n=-\infty}^{\infty} \frac{1}{\alpha_n} \begin{pmatrix} \sin \lambda_n \xi \\ -\cos \lambda_n \xi \end{pmatrix} \tilde{\varphi}_0(t, \lambda_n) - \right. \end{aligned}$$

$$-\frac{1}{\mu(\pi)} \begin{pmatrix} \sin \lambda_n^0 \xi \\ -\cos \lambda_n^0 \xi \end{pmatrix} \tilde{\varphi}_0(t, \lambda_n^0) d\xi dt = 0.$$

Using (25), we get

$$\begin{aligned} & \int_0^x (g_1^2(\mu(t)) + g_2^2(\mu(t))) \rho(t) dt + \\ & + \int_0^x \tilde{g}(\mu(t)) \rho(t) \int_0^a g(\xi) \sum_{n=-\infty}^{\infty} \frac{1}{\alpha_n} \varphi_0(\xi, \lambda_n) \tilde{\varphi}_0(t, \lambda_n) d\xi dt - \\ & - \int_0^x \tilde{g}(\mu(t)) \rho(t) \int_0^a g(\xi) \sum_{n=-\infty}^{\infty} \frac{1}{\mu(\pi)} \varphi_0(\xi, \lambda_n^0) \tilde{\varphi}_0(t, \lambda_n^0) d\xi dt + \\ & + \int_0^x \tilde{g}(\mu(t)) \rho(t) \int_0^{\alpha x - \alpha a + a} g(\xi) \sum_{n=-\infty}^{\infty} \frac{1}{\alpha_n} \varphi_0\left(\frac{\xi}{\alpha} + a - \frac{a}{\alpha}, \lambda_n\right) \tilde{\varphi}_0(t, \lambda_n) d\xi dt \\ & - \int_0^x \tilde{g}(\mu(t)) \rho(t) \int_0^{\alpha x - \alpha a + a} g(\xi) \sum_{n=-\infty}^{\infty} \frac{1}{\mu(\pi)} \varphi_0\left(\frac{\xi}{\alpha} + a - \frac{a}{\alpha}, \lambda_n^0\right) \tilde{\varphi}_0(t, \lambda_n^0) d\xi dt = 0. \end{aligned}$$

Substituting  $\frac{\xi}{\alpha} + a - \frac{a}{\alpha} \rightarrow \xi$  into the last two integrals, we obtain

$$\begin{aligned} & \int_0^x (g_1^2(\mu(t)) + g_2^2(\mu(t))) \rho(t) dt + \\ & + \int_0^x \tilde{g}(\mu(t)) \rho(t) \int_0^a g(\xi) \sum_{n=-\infty}^{\infty} \frac{1}{\alpha_n} \varphi_0(\xi, \lambda_n) \tilde{\varphi}_0(t, \lambda_n) d\xi dt - \\ & - \int_0^x \tilde{g}(\mu(t)) \rho(t) \int_0^a g(\xi) \sum_{n=-\infty}^{\infty} \frac{1}{\mu(\pi)} \varphi_0(\xi, \lambda_n^0) \tilde{\varphi}_0(t, \lambda_n^0) d\xi dt + \\ & + \alpha \int_0^x \tilde{g}(\mu(t)) \rho(t) \int_a^x g(\alpha \xi - \alpha a + a) \sum_{n=-\infty}^{\infty} \frac{1}{\alpha_n} \varphi_0(\xi, \lambda_n) \tilde{\varphi}_0(t, \lambda_n) d\xi dt - \\ & - \alpha \int_0^x \tilde{g}(\mu(t)) \rho(t) \int_a^x g(\alpha \xi - \alpha a + a) \sum_{n=-\infty}^{\infty} \frac{1}{\mu(\pi)} \varphi_0(\xi, \lambda_n^0) \tilde{\varphi}_0(t, \lambda_n^0) d\xi dt = \\ & = \int_0^x (g_1^2(\mu(t)) + g_2^2(\mu(t))) \rho(t) dt + \\ & + \int_0^x \tilde{g}(\mu(t)) \rho(t) \int_0^x g(\mu(\xi)) \rho(\xi) \sum_{n=-\infty}^{\infty} \frac{1}{\alpha_n} \varphi_0(\xi, \lambda_n) \tilde{\varphi}_0(t, \lambda_n) d\xi dt - \\ & - \int_0^x \tilde{g}(\mu(t)) \rho(t) \int_0^x g(\mu(\xi)) \rho(\xi) \sum_{n=-\infty}^{\infty} \frac{1}{\mu(\pi)} \varphi_0(\xi, \lambda_n^0) \tilde{\varphi}_0(t, \lambda_n^0) d\xi dt = 0. \quad (32) \end{aligned}$$

Using Parseval equality,

$$g(\mu(t)) = \sum_{n=-\infty}^{\infty} \left( \frac{1}{\mu(\pi)} \int_0^x g(\mu(t)) \tilde{\varphi}_0(t, \lambda_n^0) \rho(t) dt \right) \varphi_0(t, \lambda_n^0),$$

it follows from (32) that

$$\int_0^x \tilde{g}(\mu(t)) \rho(t) \int_0^x g(\mu(\xi)) \rho(\xi) \sum_{n=-\infty}^{\infty} \frac{1}{\alpha_n} \varphi_0(\xi, \lambda_n) \tilde{\varphi}_0(t, \lambda_n) d\xi dt = 0.$$

Since the system  $\{\varphi_0(t, \lambda_n)\}$ ,  $(n \in \mathbb{Z})$  is complete in  $L_{2,\rho}(0, \pi; \mathbb{C}^2)$ , we have  $g(\mu(t)) \equiv 0$ , i.e.  $(L_x g)(t) = 0$ . For  $L_x$  is invertible in  $L_2(0, \pi)$ ,  $A(x, \cdot) = 0$  is obtained.  $\square$

**Theorem 6.** *Let  $L(\Omega(x), h_1, h_2)$  and  $\hat{L}(\hat{\Omega}(x), \hat{h}_1, \hat{h}_2)$  be two boundary value problems and*

$$\lambda_n = \hat{\lambda}_n, \quad \alpha_n = \hat{\alpha}_n, \quad (n \in \mathbb{Z}).$$

*Then*

$$\Omega(x) = \hat{\Omega}(x) \text{ a.e. on } (0, \pi), \quad h_1 = \hat{h}_1, \quad h_2 = \hat{h}_2.$$

*Proof.* According to (18) and (19),  $F_0(x, t) = \hat{F}_0(x, t)$  and  $F(x, t) = \hat{F}(x, t)$ . Then, from the fundamental equation (17), we have  $A(x, t) = \hat{A}(x, t)$ . It follows from (5) that  $\Omega(x) = \hat{\Omega}(x)$  a.e. on  $(0, \pi)$ . Taking into account (4), we find  $\varphi(x, \lambda_n) = \hat{\varphi}(x, \lambda_n)$ . In consideration of (14), we get  $\dot{\Delta}(\lambda_n) = \dot{\hat{\Delta}}(\lambda_n)$  and from (8),  $\beta_n = \hat{\beta}_n$ . Thus, using (3) and (7), we obtain  $h_1 = \hat{h}_1$ ,  $h_2 = \hat{h}_2$ .  $\square$

## 5. SOLUTION OF INVERSE PROBLEM

Let the real numbers  $\{\lambda_n, \alpha_n\}$ ,  $(n \in \mathbb{Z})$  of the form (10) and (11) be given. Using these numbers, we construct the functions  $F_0(x, t)$  and  $F(x, t)$  by the formulas (18) and (19) and determine  $A(x, t)$  from the fundamental equation (17).

Now, let us construct the function  $\varphi(x, \lambda)$  by the formula (4), the function  $\Omega(x)$  by the formula (5),  $\Delta(\lambda)$  by the formula (14) and  $\beta_n$  by the formula (8) respectively, i.e.,

$$\varphi(x, \lambda) := \varphi_0(x, \lambda) + \int_0^{\mu(x)} A(x, t) \begin{pmatrix} \sin \lambda t \\ -\cos \lambda t \end{pmatrix} dt,$$

$$\Omega(x) := \rho(x)A(x, \mu(x))B - BA(x, \mu(x)),$$

$$\Delta(\lambda) := -\mu(\pi)(\lambda_0^2 - \lambda^2) \prod_{n=1}^{\infty} \frac{\lambda_n^2 - \lambda^2}{(\lambda_n^0)^2},$$

$$\beta_n := \frac{\dot{\Delta}(\lambda_n)}{\alpha_n} \neq 0.$$

The function  $F_0(x, t)$  can be rewritten as follows:

$$F_0(x, t) = \frac{1}{2} [a(x - \mu(t)) + a(x + \mu(t))T],$$

where

$$a(x) = \sum_{n=-\infty}^{\infty} \left[ \frac{1}{\alpha_n} \begin{pmatrix} \cos \lambda_n x & -\sin \lambda_n x \\ \sin \lambda_n x & \cos \lambda_n x \end{pmatrix} - \frac{1}{\mu(\pi)} \begin{pmatrix} \cos \lambda_n^0 x & -\sin \lambda_n^0 x \\ \sin \lambda_n^0 x & \cos \lambda_n^0 x \end{pmatrix} \right]$$

and  $T = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ . Analogously in [7], it is shown that the function  $a(x) \in W_2^1[0, 2\pi]$ .

### 5.1. Derivation of the Differential Equation.

**Lemma 7.** *The relations hold:*

$$B\varphi'(x, \lambda) + \Omega(x)\varphi(x, \lambda) = \lambda\rho(x)\varphi(x, \lambda), \quad (33)$$

$$\varphi_1(0, \lambda) = 0, \quad \varphi_2(0, \lambda) = -1. \quad (34)$$

*Proof.* Differentiating on  $x$  and  $y$  the equation (17) respectively, we get

$$A'_x(x, \mu(t)) + F'_x(x, t) + A(x, \mu(x))F_0(\mu(x), t) + \int_0^{\mu(x)} A'_x(x, \xi)F_0(\xi, t)d\xi = 0, \quad (35)$$

$$\rho(t)A'_t(x, \mu(t)) + F'_t(x, t) + \int_0^{\mu(x)} A(x, \xi)F'_{0t}(\xi, t)d\xi = 0. \quad (36)$$

It follows from (18) and (19) that

$$\frac{\partial}{\partial t}F_0(x, t)B + \rho(t)B\frac{\partial}{\partial x}F_0(x, t) = 0, \quad (37)$$

$$\rho(x)\frac{\partial}{\partial t}F(x, t)B + \rho(t)B\frac{\partial}{\partial x}F(x, t) = 0. \quad (38)$$

Since  $F_0(x, 0)BS = 0$  and  $F(x, 0)BS = 0$ , where  $S = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$ , using the main equation (17), we obtain

$$A(x, 0)BS = 0, \quad (39)$$

or

$$A_{11}(x, 0) = A_{21}(x, 0) = 0.$$

Multiplying the equation (35) on the left by  $B$  and  $\rho(t)$  we get

$$\begin{aligned} &\rho(t)BF'_x(x, t) + \rho(t)BA'_x(x, \mu(t)) + \rho(t)BA(x, \mu(x))F_0(\mu(x), t) + \\ &+ \rho(t) \int_0^{\mu(x)} BA'_x(x, \xi)F_0(\xi, t)d\xi = 0 \end{aligned} \quad (40)$$

and multiplying the equation (36) on the right by  $B$  and  $\rho(x)$  we have

$$\rho(x)F'_t(x, t)B + \rho(x)\rho(t)A'_t(x, \mu(t))B + \rho(x) \int_0^{\mu(x)} A(x, \xi)F'_{0t}(\xi, t)Bd\xi = 0. \quad (41)$$

Adding (40) and (41) and using (38), we find

$$\begin{aligned} &\rho(t)BA'_x(x, \mu(t)) + \rho(t)BA(x, \mu(x))F_0(\mu(x), t) + \rho(t) \int_0^{\mu(x)} BA'_x(x, \xi)F_0(\xi, t)d\xi = \\ &= -\rho(x)\rho(t)A'_t(x, \mu(t))B - \rho(x) \int_0^{\mu(x)} A(x, \xi)F'_{0t}(\xi, t)Bd\xi \equiv I(x, t). \end{aligned} \quad (42)$$

From (37), we get

$$I(x, t) = -\rho(x)\rho(t)A'_t(x, \mu(t))B + \rho(x)\rho(t) \int_0^{\mu(x)} A(x, \xi)BF'_{0\xi}(\xi, t)d\xi. \quad (43)$$

Integrating by parts and from (39)

$$\begin{aligned} I(x, t) &= -\rho(x)\rho(t)A'_t(x, \mu(t))B + \rho(t)\rho(x)A(x, \mu(x))BF_0(\mu(x), t) - \\ &- \rho(x)\rho(t) \int_0^{\mu(x)} A'_\xi(x, \xi)BF_0(\xi, t)d\xi \end{aligned} \quad (44)$$

is obtained. Substituting (44) into (42) and divided by  $\rho(t) \neq 0$ , we have

$$BA'_x(x, \mu(t)) + BA(x, \mu(x))F_0(\mu(x), t) - \rho(x)A(x, \mu(x))BF_0(\mu(x), t) + \\ + \rho(x)A'_t(x, \mu(t))B + \int_0^{\mu(x)} \left[ BA'_x(x, \xi) + \rho(x)A'_\xi(x, \xi)B \right] F_0(\xi, t) d\xi = 0. \quad (45)$$

Multiplying (17) on the left by  $\Omega(x)$  in the form of (5) and add to (45)

$$BA'_x(x, \mu(t)) + \rho(x)A'_t(x, \mu(t))B + \Omega(x)A(x, \mu(t)) + \\ + \int_0^{\mu(x)} \left[ BA'_x(x, \xi) + \rho(x)A'_\xi(x, \xi)B + \Omega(x)A(x, \xi) \right] F_0(\xi, t) dt = 0 \quad (46)$$

is obtained. Setting

$$J(x, t) := BA'_x(x, t) + \rho(x)A'_t(x, t)B + \Omega(x)A(x, t),$$

we can rewrite equation (46) as follows

$$J(x, \mu(t)) + \int_0^{\mu(x)} J(x, \xi) F_0(\xi, t) d\xi = 0. \quad (47)$$

According to Lemma 5, homogeneous equation (47) has only the trivial solution, i.e.

$$BA'_x(x, t) + \rho(x)A'_t(x, t)B + \Omega(x)A(x, t) = 0, \quad 0 < t < x. \quad (48)$$

Differentiating (4) and multiplying on the left by  $B$ , we have

$$B\varphi'(x, \lambda) = \lambda\rho(x)B \begin{pmatrix} \cos \lambda\mu(x) \\ \sin \lambda\mu(x) \end{pmatrix} + BA(x, \mu(x)) \begin{pmatrix} \sin \lambda\mu(x) \\ -\cos \lambda\mu(x) \end{pmatrix} + \\ + \int_0^{\mu(x)} BA'_x(x, t) \begin{pmatrix} \sin \lambda t \\ -\cos \lambda t \end{pmatrix} dt. \quad (49)$$

On the other hand, multiplying (4) on the left by  $\lambda\rho(x)$  and then integrating by parts and using (39), we find

$$\lambda\rho(x)\varphi(x, \lambda) = \lambda\rho(x) \begin{pmatrix} \sin \lambda\mu(x) \\ -\cos \lambda\mu(x) \end{pmatrix} + \rho(x)A(x, \mu(x))B \begin{pmatrix} \sin \lambda\mu(x) \\ -\cos \lambda\mu(x) \end{pmatrix} - \\ - \rho(x) \int_0^{\mu(x)} A'_t(x, t)B \begin{pmatrix} \sin \lambda t \\ -\cos \lambda t \end{pmatrix} dt. \quad (50)$$

It follows from (49) and (50) that

$$\lambda\rho(x)\varphi(x, \lambda) = B\varphi'(x, \lambda) - [BA(x, \mu(x)) - \rho(x)A(x, \mu(x))B] \begin{pmatrix} \sin \lambda\mu(x) \\ -\cos \lambda\mu(x) \end{pmatrix} - \\ - \int_0^{\mu(x)} \left[ BA'_x(x, t) + \rho(x)A'_t(x, t)B \right] \begin{pmatrix} \sin \lambda t \\ -\cos \lambda t \end{pmatrix} dt.$$

Taking into account (5) and (48),

$$B\varphi'(x, \lambda) + \Omega(x)\varphi(x, \lambda) = \lambda\rho(x)\varphi(x, \lambda)$$

is obtained. For  $x = 0$ , from (4) we get (34).  $\square$

### 5.2. Derivation of Parseval Equality.

**Lemma 8.** *For each function  $g(x) \in L_{2,\rho}(0, \pi; \mathbb{C}^2)$ , the following relation holds:*

$$\int_0^\pi (g_1^2(x) + g_2^2(x)) \rho(x) dx = \sum_{n=-\infty}^{\infty} \frac{1}{\alpha_n} \left( \int_0^\pi \tilde{\varphi}(t, \lambda_n) g(t) \rho(t) dt \right)^2. \quad (51)$$

*Proof.* It follows from (4) and (25) that

$$\varphi(x, \lambda) = \varphi_0(x, \lambda) + \int_0^x A(x, \mu(t)) \varphi_0(t, \lambda) \rho(t) dt. \quad (52)$$

Using the expression

$$F_0(x, t) = \begin{cases} F(x, t), & x < a, \\ F\left(\frac{x}{\alpha} + a - \frac{a}{\alpha}, t\right), & x > a, \end{cases}$$

main equation (17) transforms into the following form

$$A(x, \mu(t)) + F(x, t) + \int_0^x A(x, \mu(\xi)) F(\xi, t) \rho(\xi) d\xi = 0. \quad (53)$$

From the relation (52), we get

$$\varphi_0(x, \lambda) = \varphi(x, \lambda) + \int_0^x H(x, \mu(t)) \varphi(t, \lambda) \rho(t) dt \quad (54)$$

and for the kernel  $H(x, \mu(t))$  we have the identity

$$\tilde{H}(x, \mu(t)) = F(t, x) + \int_0^x A(x, \mu(\xi)) F(\xi, t) \rho(\xi) d\xi. \quad (55)$$

Denote

$$Q(\lambda) := \int_0^\pi \tilde{\varphi}(t, \lambda) g(t) \rho(t) dt$$

and using (52) transform into the following form

$$Q(\lambda) = \int_0^\pi \tilde{\varphi}_0(t, \lambda) h(t) \rho(t) dt,$$

where

$$h(t) = g(t) + \int_t^\pi \tilde{A}(s, \mu(t)) g(s) \rho(s) ds. \quad (56)$$

Similarly, in view of (54), we have

$$g(t) = h(t) + \int_t^\pi \tilde{H}(s, \mu(t)) h(s) \rho(s) ds. \quad (57)$$

According to (56),

$$\begin{aligned} \int_0^\pi F(x, t) h(t) \rho(t) dt &= \int_0^\pi F(x, t) \left[ g(t) + \int_t^\pi \tilde{A}(s, \mu(t)) g(s) \rho(s) ds \right] \rho(t) dt \\ &= \int_0^\pi \left[ F(x, t) + \int_0^t F(x, s) \tilde{A}(t, \mu(s)) \rho(s) ds \right] g(t) \rho(t) dt \\ &= \int_0^x \left[ F(x, t) + \int_0^t F(x, s) \tilde{A}(t, \mu(s)) \rho(s) ds \right] g(t) \rho(t) dt + \\ &\quad + \int_x^\pi \left[ F(x, t) + \int_0^t F(x, s) \tilde{A}(t, \mu(s)) \rho(s) ds \right] g(t) \rho(t) dt. \end{aligned}$$

It follows from (53) and (55) that

$$\int_0^\pi F(x, t)h(t)\rho(t)dt = \int_0^x H(x, \mu(t))g(t)\rho(t)dt - \int_x^\pi \tilde{A}(t, \mu(x))g(t)\rho(t)dt. \quad (58)$$

From (22) and Parseval equality we obtain,

$$\begin{aligned} & \int_0^\pi (h_1^2(t) + h_2^2(t)) \rho(t)dt + \int_0^\pi \tilde{h}(x)F(x, t)h(t)\rho(t)\rho(x)dxdt = \\ &= \int_0^\pi (h_1^2(t) + h_2^2(t)) \rho(t)dt + \sum_{n=-\infty}^\infty \frac{1}{\alpha_n} \left( \int_0^\pi \tilde{\varphi}_0(t, \lambda_n)h(t)\rho(t)dt \right)^2 \\ & \quad - \sum_{n=-\infty}^\infty \frac{1}{\mu(\pi)} \left( \int_0^\pi \tilde{\varphi}_0(t, \lambda_n^0)h(t)\rho(t)dt \right)^2 \\ &= \sum_{n=-\infty}^\infty \frac{1}{\alpha_n} \left( \int_0^\pi \tilde{\varphi}_0(t, \lambda_n)h(t)\rho(t)dt \right)^2 = \sum_{n=-\infty}^\infty \frac{Q^2(\lambda_n)}{\alpha_n}. \end{aligned}$$

Taking into account (58), we have

$$\begin{aligned} & \sum_{n=-\infty}^\infty \frac{Q^2(\lambda_n)}{\alpha_n} = \int_0^\pi (h_1^2(t) + h_2^2(t)) \rho(t)dt + \\ & \quad + \int_0^\pi \tilde{h}(x) \left( \int_0^x H(x, \mu(t))g(t)\rho(t)dt \right) \rho(x)dx - \\ & \quad - \int_0^\pi \tilde{h}(x) \left( \int_x^\pi \tilde{A}(t, \mu(x))g(t)\rho(t)dt \right) \rho(x)dx \\ &= \int_0^\pi (h_1^2(t) + h_2^2(t)) \rho(t)dt + \int_0^\pi \left( \int_t^\pi \tilde{h}(x)H(x, \mu(t))\rho(x)dx \right) g(t)\rho(t)dt - \\ & \quad - \int_0^\pi \tilde{h}(x) \left( \int_x^\pi \tilde{A}(t, \mu(x))g(t)\rho(t)dt \right) \rho(x)dx, \end{aligned}$$

whence by formulas (56) and (57),

$$\begin{aligned} & \sum_{n=-\infty}^\infty \frac{Q^2(\lambda_n)}{\alpha_n} = \int_0^\pi (h_1^2(t) + h_2^2(t)) \rho(t)dt + \\ & \quad + \int_0^\pi (\tilde{g}(t) - \tilde{h}(t)) g(t)\rho(t)dt - \int_0^\pi \tilde{h}(x) (h(x) - g(x)) \rho(x)dx \\ &= \int_0^\pi (g_1^2(t) + g_2^2(t)) \rho(t)dt \end{aligned}$$

is obtained, i.e., the relation (51) is valid.  $\square$

**Corollary 9.** For any function  $f(x)$  and  $g(x) \in L_{2,\rho}(0, \pi; \mathbb{C}^2)$ , the relation holds:

$$\int_0^\pi \tilde{g}(x)f(x)\rho(x)dx = \sum_{n=-\infty}^\infty \frac{1}{\alpha_n} \left( \int_0^\pi \tilde{g}(t)\varphi(t, \lambda_n)\rho(t)dt \right) \left( \int_0^\pi \tilde{\varphi}(t, \lambda_n)f(t)\rho(t)dt \right). \quad (59)$$

**Lemma 10.** *For any  $f(x) \in W_2^1[0, \pi]$ , the expansion formula*

$$f(x) = \sum_{n=-\infty}^{\infty} c_n \varphi(x, \lambda_n) \quad (60)$$

is valid, where

$$c_n = \frac{1}{\alpha_n} \int_0^\pi \tilde{\varphi}(x, \lambda_n) f(x) \rho(x) dx.$$

*Proof.* Consider the series

$$f^*(x) = \sum_{n=-\infty}^{\infty} c_n \varphi(x, \lambda_n), \quad (61)$$

where

$$c_n := \frac{1}{\alpha_n} \int_0^\pi \tilde{\varphi}(x, \lambda_n) f(x) \rho(x) dx. \quad (62)$$

Using Lemma 7 and integrating by parts, we get

$$\begin{aligned} c_n &= \frac{1}{\alpha_n \lambda_n} \int_0^\pi \left[ -\frac{\partial}{\partial x} \tilde{\varphi}(x, \lambda_n) B + \tilde{\varphi}(x, \lambda_n) \Omega(x) \right] f(x) dx = \\ &= \frac{-1}{\alpha_n \lambda_n} [\tilde{\varphi}(\pi, \lambda_n) B f(\pi) - \tilde{\varphi}(0, \lambda_n) B f(0)] + \\ &\quad + \frac{1}{\alpha_n \lambda_n} \int_0^\pi \tilde{\varphi}(x, \lambda_n) [B f'(x) + \Omega(x) f(x)] dx. \end{aligned}$$

Applying the asymptotic formulas in Theorem 1, we find  $\{c_n\} \in l_2$ . Consequently the series (61) converges absolutely and uniformly on  $[0, \pi]$ . According to (59) and (62), we have

$$\begin{aligned} &\int_0^\pi \tilde{g}(x) f(x) \rho(x) dx = \\ &= \sum_{n=-\infty}^{\infty} \frac{1}{\alpha_n} \left( \int_0^\pi \tilde{g}(t) \varphi(t, \lambda_n) \rho(t) dt \right) \left( \int_0^\pi \tilde{\varphi}(t, \lambda_n) f(t) \rho(t) dt \right) \\ &= \sum_{n=-\infty}^{\infty} c_n \left( \int_0^\pi \tilde{g}(t) \varphi(t, \lambda_n) \rho(t) dt \right) = \int_0^\pi \tilde{g}(t) \left( \sum_{n=-\infty}^{\infty} c_n \varphi(t, \lambda_n) \right) \rho(t) dt \\ &= \int_0^\pi \tilde{g}(t) f^*(t) \rho(t) dt. \end{aligned}$$

Since  $g(x)$  is arbitrary,  $f(x) = f^*(x)$  is obtained, i.e., the expansion formula (60) is found.  $\square$

### 5.3. Derivation of Boundary Condition.

**Lemma 11.** *The following equality holds:*

$$\sum_{n=-\infty}^{\infty} \frac{\varphi(x, \lambda_n)}{\alpha_n \beta_n} = 0. \quad (63)$$



*Proof.* Using residue theorem, we get

$$\sum_{n=-\infty}^{\infty} \frac{\varphi(x, \lambda_n)}{\alpha_n \beta_n} = \sum_{n=-\infty}^{\infty} \frac{\varphi(x, \lambda_n)}{\dot{\Delta}(\lambda_n)} = \sum_{n=-\infty}^{\infty} \operatorname{Res}_{\lambda=\lambda_n} \frac{\varphi(x, \lambda)}{\Delta(\lambda)} = \frac{1}{2\pi i} \int_{\Gamma_N} \frac{\varphi(x, \lambda)}{\Delta(\lambda)} d\lambda, \quad (64)$$

where  $\Gamma_N = \left\{ \lambda : |\lambda| = \frac{N\pi}{\mu(\pi)} + \frac{\pi}{2\mu(\pi)} \right\}$ . From (14) and ([19], Lemma 3.4.2),

$$\Delta(\lambda) = \lambda \sin \lambda \mu(\pi) + O(e^{|Im \lambda| \mu(\pi)}). \quad (65)$$

We denote  $G_\delta = \left\{ \lambda : \left| \lambda - \frac{n\pi}{\mu(\pi)} \right| \geq \delta, \quad n = 0, \pm 1, \pm 2, \dots \right\}$  for some small fixed  $\delta > 0$  and  $|\sin \lambda \mu(\pi)| \geq C_\delta e^{|Im \lambda| \mu(\pi)}$ ,  $\lambda \in G_\delta$ , where  $C_\delta$  positive number. Therefore, we have

$$|\Delta(\lambda)| \geq C_\delta |\lambda| e^{|Im \lambda| \mu(\pi)}, \quad \lambda \in G_\delta.$$

Using this inequality and (12), we obtain (63).  $\square$

**Theorem 12.** *The following relation is valid:*

$$(\lambda_n + h_1) \varphi_1(\pi, \lambda_n) + h_2 \varphi_2(\pi, \lambda_n) = 0.$$

*Proof.* From (63), we can write for any  $n_0 \in \mathbb{Z}$

$$\frac{\varphi(x, \lambda_{n_0})}{\alpha_{n_0}} = - \sum_{\substack{n=-\infty \\ n \neq n_0}}^{\infty} \frac{\beta_{n_0} \varphi(x, \lambda_n)}{\alpha_n \beta_n} \quad (66)$$

Let  $m \neq n_0$  be any fixed number and  $f(x) = \varphi(x, \lambda_k)$ . Then substituting (66) in (60)

$$\varphi(x, \lambda_k) = \sum_{\substack{n=-\infty \\ n \neq n_0}}^{\infty} c_{nk} \varphi(x, \lambda_n),$$

where

$$c_{nk} = \frac{1}{\alpha_n} \int_0^\pi \left[ \tilde{\varphi}(t, \lambda_n) - \frac{\beta_{n_0}}{\beta_n} \tilde{\varphi}(t, \lambda_{n_0}) \right] \varphi(t, \lambda_k) \rho(t) dt.$$

The system of functions  $\{\varphi_0(x, \lambda_n)\}$ ,  $(n \in \mathbb{Z})$  is orthogonal in  $L_{2,\rho}(0, \pi; \mathbb{C}^2)$ . Then by (4), the system of functions  $\{\varphi(x, \lambda_n)\}$ ,  $(n \in \mathbb{Z})$  is orthogonal in  $L_{2,\rho}(0, \pi; \mathbb{C}^2)$  as well. Therefore,  $c_{nk} = \delta_{nk}$ , where  $\delta_{nk}$  is Kronecker delta. Let us define

$$a_{nk} := \int_0^\pi \tilde{\varphi}(t, \lambda_n) \varphi(t, \lambda_k) \rho(t) dt. \quad (67)$$

Using this expression, we have for  $n \neq k$

$$a_{kk} - \frac{\beta_n}{\beta_k} a_{nk} = \alpha_k. \quad (68)$$

It follows from (67) that  $a_{nk} = a_{kn}$ . Taking into account this equality and (68),

$$\beta_k^2 (\alpha_k - a_{kk}) = \beta_n^2 (\alpha_n - a_{nn}) = H, \quad k \neq n,$$

where  $H$  is a constant. Then, we have

$$\int_0^\pi \tilde{\varphi}(t, \lambda_n) \varphi(t, \lambda_n) \rho(t) dt = \alpha_n - \frac{H}{\beta_n^2}$$

and

$$\int_0^\pi \tilde{\varphi}(t, \lambda_k) \varphi(t, \lambda_n) \rho(t) dt = -\frac{H}{\beta_k \beta_n}, \quad k \neq n.$$

It is easily obtained that for  $k \neq n$ ,

$$\begin{aligned} & \int_0^\pi [\varphi_1(x, \lambda_k) \varphi_1(x, \lambda_n) + \varphi_2(x, \lambda_k) \varphi_2(x, \lambda_n)] \rho(x) dx = \\ & = \frac{1}{(\lambda_k - \lambda_n)} [\varphi_2(\pi, \lambda_k) \varphi_1(\pi, \lambda_n) - \varphi_1(\pi, \lambda_k) \varphi_2(\pi, \lambda_n)] = -\frac{H}{\beta_k \beta_n}. \end{aligned}$$

According to the last equation, for  $n \neq k$ ,

$$\beta_k \varphi_2(\pi, \lambda_k) \beta_n \varphi_1(\pi, \lambda_n) - \beta_k \varphi_1(\pi, \lambda_k) \beta_n \varphi_2(\pi, \lambda_n) = -H(\lambda_k - \lambda_n). \quad (69)$$

We denote

$$D_n := \beta_n \varphi_1(\pi, \lambda_n), \quad E_n := \beta_n \varphi_2(\pi, \lambda_n). \quad (70)$$

Then, we can rewrite equation (69) as follows

$$D_k E_n - E_k D_n = H(\lambda_k - \lambda_n), \quad n \neq k. \quad (71)$$

Let  $i, j, k, n$  be pairwise distinct integers, then we get

$$\begin{aligned} D_k E_n - E_k D_n &= H(\lambda_k - \lambda_n), \\ D_n E_i - E_n D_i &= H(\lambda_n - \lambda_i), \\ D_i E_k - E_i D_k &= H(\lambda_i - \lambda_k). \end{aligned}$$

Adding them together, we find

$$D_n(E_i - E_k) + E_n(D_k - D_i) = E_i D_k - D_i E_k.$$

In this equation, replacing  $n$  by  $j$ , we get another equation

$$D_j(E_i - E_k) + E_j(D_k - D_i) = E_i D_k - D_i E_k.$$

Subtracting the last two equation,

$$(D_n - D_j)(E_i - E_k) = (D_i - D_k)(E_n - E_j).$$

In the case of  $E_n = E_j$ , for some  $n, j \in \mathbb{Z}$ , then  $E_n = \text{const}$ . From (71),  $D_n = \kappa_1 \lambda_n + \kappa_2$ . In the case of  $E_n \neq E_j$ , then we obtain  $D_n = \kappa_1 \lambda_n + \kappa_2$  and  $E_n = \kappa_3 \lambda_n + \kappa_4$ , where in both cases  $\kappa_1, \kappa_2, \kappa_3, \kappa_4$  are constant. Therefore, using these relation in (70), we find

$$\beta_n \varphi_1(\pi, \lambda_n) = \kappa_1 \lambda_n + \kappa_2, \quad \beta_n \varphi_2(\pi, \lambda_n) = \kappa_3 \lambda_n + \kappa_4.$$

Using

$$\varphi_1(\pi, \lambda_n) = O\left(\frac{1}{n}\right), \quad \varphi_2(\pi, \lambda_n) = (-1)^{n+1} + O\left(\frac{1}{n}\right),$$

$\lambda_n = \frac{n\pi}{\mu(\pi)} + O\left(\frac{1}{n}\right)$  and  $\beta_n = \frac{n\pi}{\mu(\pi)}(-1)^n + O(1)$  derived from (8) and (65), we obtain  $\kappa_1 = 0, \kappa_3 = -1$ . Denoting  $h_2 := \kappa_2$  and  $h_1 := -\kappa_4$ ,

$$h_2 \varphi_2(\pi, \lambda_n) = -(\lambda_n + h_1) \varphi_1(\pi, \lambda_n), \quad n \in \mathbb{Z}$$

is obtained and it follows from (71) that  $H = h_2$ .  $\square$

Thus, we have proved the following theorem:

**Theorem 13.** *For the sequences  $\{\lambda_n, \alpha_n\}$ ,  $(n \in \mathbb{Z})$ , to be the spectral data for a certain boundary value problem  $L(\Omega(x), h_1, h_2)$  of the form (1), (2) with  $\Omega(x) \in L_2(0, \pi)$  it is necessary and sufficient that the relations (9) and (11) hold.*

Algorithm of the construction of the function  $\Omega(x)$  by spectral data  $\{\lambda_n, \alpha_n\}$ , ( $n \in \mathbb{Z}$ ) follows from the proof of the theorem:

- 1) By the given numbers  $\{\lambda_n, \alpha_n\}$ , ( $n \in \mathbb{Z}$ ) the functions  $F_0(x, t)$  and  $F(x, t)$  are respectively constructed by formula (18) and (19),
- 2) The function  $A(x, t)$  is found from equation (17),
- 3)  $\Omega(x)$  is calculated by the formula (5).

**Acknowledgment.** This work is supported by The Scientific and Technological Research Council of Turkey (TÜBİTAK).

#### REFERENCES

- [1] M. J. Ablowitz, D. J. Kaup, A. C. Newell and H. Segur, *Nonlinear-evolution equations of physical significance*, Phys. Rev. Lett. 31 (1973), 125-127.
- [2] A. Çöl and Kh. R. Mamedov, *On an inverse scattering problem for a class of Dirac operators with spectral parameter in the boundary condition*, J. Math. Anal. Appl. 393 (2012), 470-478.
- [3] S. Albeverio, R. Hryniv and Ya. Mykytyuk, *Inverse spectral problems for Dirac operators with summable potentials*, Russ. J. Math. Phys. 12(14) (2005), 406-423.
- [4] O. Aydoğdu, E. Maghsoodi and H. Hassanabadi, *Dirac equation for the Hulthen potential within the Yukawa-type tensor interaction*, Chin. Phys. B 22(1) (2013), 010302.
- [5] O. Aydoğdu and R. Sever, *Pseudospin and spin symmetry in Dirac-Morse problem with a tensor potential*, Physics Letters B 703(3) (2011), 379-385.
- [6] T. T. Dzabiev, *The inverse problem for the Dirac equation with a singularity*, Dokl. Akad. Nauk Azerbaidzan SSR 22(11) (1966), 8-12.
- [7] G. Freiling and V. Yurko, *Inverse Sturm-Liouville Problems and Their Applications*, Nova Science Publisher, Huntington, New York, 2008.
- [8] B. Fritzsche, B. Kirstein and A. L. Sakhnovich, *Semiseparable integral operators and explicit solution of an inverse problem for a skew-self-adjoint Dirac-type system*, Integral Equations and Operator Theory, 66(2) (2010), 231-251.
- [9] B. Fritzsche, B. Kirstein, I.Ya. Roitberg and A.L. Sakhnovich, *Skew-self-adjoint Dirac system with a rectangular matrix potential: Weyl theory, direct and inverse problems*, Integral Equations and Operator Theory 74(2) (2012), 163-187.
- [10] M. G. Gasymov and T. T. Dzabiev, *Solution of the inverse problem by two spectra for the Dirac equation on a finite interval*, Dokl. Akad. Nauk Azerbaidzan SSR 22(7) (1966), 3-6.
- [11] M. G. Gasymov, *The inverse scattering problem for a system of Dirac equations of order  $2n$* , Trans. Moskow Math. Soc. 19 (1968), 41-119.
- [12] M. Horvath, *On a theorem of Ambarzumian*, Proc. Roy. Soc. Edinburg Sect. A 131(4) (2001), 899-907.
- [13] M. Kiss, *An  $n$ -dimensional Ambarzumian type theorem for Dirac operators*, Inverse Problems 20 (2004), 1593-1597.
- [14] A. R. Latifova, *On the representation of solution with initial conditions for Dirac equations system with discontinuous coefficient*, Proceeding of IMM of NAS of Azerbaijan 16(24) (2002), 64-66.
- [15] B. Ya. Levin, *Lectures on Entire Functions*, American Mathematical Society, Providence, 1996.
- [16] S.G. Mamedov, *The inverse boundary value problem on a finite interval for Dirac's system of equations*, Azerbaidzan Gos. Univ. Ucen. Zap. Ser. Fiz-Mat. Nauk 5 (1975), 61-67.
- [17] Kh. R. Mamedov and A. Çöl, *On an inverse scattering problem for a class Dirac operator with discontinuous coefficient and nonlinear dependence on the spectral parameter in the boundary condition*, Mathematical Methods in the Applied Sciences 35(14) (2012), 1712-1720.
- [18] Kh. R. Mamedov and Ö. Akçay, *Inverse problem for a class of Dirac operator*, Taiwanese Journal of Mathematics 18(3) (2014), 753-772.
- [19] V. A. Marchenko, *Sturm-Liouville Operators and Applications*, AMS Chelsea Publishing, Providence, Rhode Island, 2011.
- [20] A. B. de Monvel and D. Shepelsky, *Initial boundary value problem for the mKdV equation on a finite interval*, Ann. Inst. Fourier, Grenoble 54(5) (2004), 1477-1495.

- [21] Ya. V. Mykytyuk and D. V. Puyda, *Inverse spectral problems for Dirac operators on a finite interval*, J. Math. Anal. Appl. 386(1) (2012), 177–194.
- [22] I. M. Nabiev, *Solution of the inverse quasiperiodic problem for the Dirac system*, Mathematical Notes 89(5-6) (2011), 845–852.
- [23] E. S. Panakhov, *Some aspects inverse problem for Dirac operator with peculiarity*, Transactions of NAS of Azerbaijan 3 (1995), 39–44.
- [24] D. V. Puyda, *Inverse spectral problems for Dirac operators with summable matrix-valued potentials*, Integral Equations and Operator Theory, 74(3) (2012), 417–450.
- [25] A. Sakhnovich, *Skew-self-adjoint discrete and continuous Dirac-type systems: inverse problems and Borg-Marchenko theorems*, Inverse Problems 22(6) (2006), 2083–2101.
- [26] D. G. Shepelsky, *An inverse spectral problem for a Dirac-type operator with sewing*, Dynamical Systems and Complex Analysis, Naukova Dumka, Kiev, 1992, pp. 104–112 (Russian).
- [27] B. Thaller, *The Dirac Equation*, Springer, Berlin, 1992.
- [28] B. A. Watson, *Inverse spectral problems for weighted Dirac systems*, Inverse Problems 15 (1999), 793–805.
- [29] C.-Fu Yang and Z.-You Huang, *Reconstruction of the Dirac operator from nodal data*, Integral Equations and Operator Theory 66(4) (2010), 539–551.
- [30] C.-Fu Yang and X.-Ping Yang, *Some Ambarzumian-type theorems for Dirac operators*, Inverse Problems 25(9) (2009), 095012 (13pp).

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